

3. INDUCTION AND FINITE SERIES

§3.1. Mathematical Induction

Often we note a pattern and we have a hunch that it will always be the case. For example:

$$1^3 + 2^3 = 1 + 8 = 9 = 3^2 = (1 + 2)^2;$$

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = 6^2 = (1 + 2 + 3)^2;$$

$$1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 = 10^2 = (1 + 2 + 3 + 4)^2.$$

Is this a coincidence, or will this pattern continue forever?

A **prime number** is one that is bigger than 1 and has no factors other than 1 and itself. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ... Now notice this pattern.

The number $0^2 + 0 + 41 = 41$, which is a prime number.

The number $1^2 + 1 + 41 = 43$, which is a prime number.

The number $2^2 + 2 + 41 = 47$, which is a prime number.

The number $3^2 + 3 + 41 = 53$, which is a prime number.

The number $4^2 + 4 + 41 = 61$, which is a prime number.

This is more than a coincidence surely. Let's make sure.

The number $5^2 + 5 + 41 = 71$, which is a prime number.

The number $6^2 + 6 + 41 = 83$, which is a prime number.

The number $7^2 + 7 + 41 = 97$, which is a prime number.

The number $8^2 + 8 + 41 = 113$, which is a prime number.
The number $9^2 + 9 + 41 = 131$, which is a prime number.
The number $10^2 + 10 + 41 = 151$, which is a prime number.

If this was a scientific enquiry then we could report that we carried out an experiment eleven times and got the same result every time. But this is mathematics. Even a large number of instances of a pattern does not guarantee that the pattern will go on forever.

If you tried this out for thirty more numbers you'd still reach the same conclusion. And yet a moment's reflection will tell you that when we reach $41^2 + 41 + 41$ the number will be divisible by 41 and hence won't be prime. In fact $40^2 + 40 + 41$ is also divisible by 41. But for $n = 1$ to 39, $n^2 + n + 41$ is prime.

To guarantee that something *always* works in mathematics we need a **proof**. Mathematical induction is a technique for proving that a certain pattern works for *all* n . It's a very simple principle that comes in two stages. We firstly verify that it works for $n = 1$. Then we assume that it's true for n and then prove that it's true for $n + 1$. It's like a ladder. You can get to any rung because there is a way of getting from each rung to the next. But the bottom of the ladder may be suspended 3 metres above ground. You have to make sure that you can get to the bottom rung.

Theorem 1 (Principle of Induction): Suppose $P(n)$ is some statement that includes a variable n .

Suppose that:

- (1) $P(1)$ is true and
- (2) If $P(n)$ is true then $P(n + 1)$ is true.

Then $P(n)$ is true for all positive integers n .

Proof: By (1) $P(1)$ is true.

Hence by (2) $P(2)$ is true.

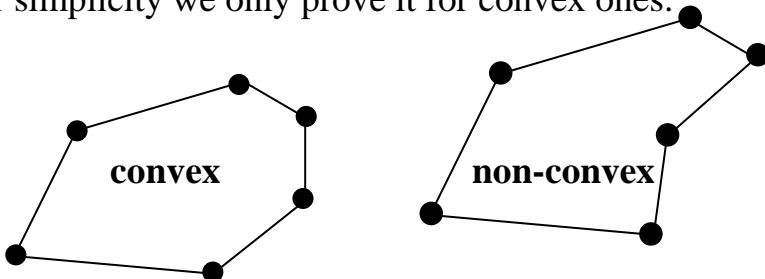
Hence by (3) $P(3)$ is true.

..... 🙌😊



It's not necessary to start at $n = 1$. Often we'll start at $n = 0$. There may be special circumstances that require us to start at $n = 7$. Wherever we start the conclusion will be that the statement is true for all values of n from the starting value onwards.

A polygon is a region of the plane that is bounded by straight edges. A convex polygon is one where the line joining any two vertices lies within the polygon. The following theorem works for non-convex polygons but for simplicity we only prove it for convex ones.



Example 1: Prove that for all $n \geq 3$ the sum of the interior angles of a convex n -sided polygon is $180n - 360$ degrees.

Solution: Here the induction begins at $n = 3$ because a polygon can't have fewer than 3 sides.

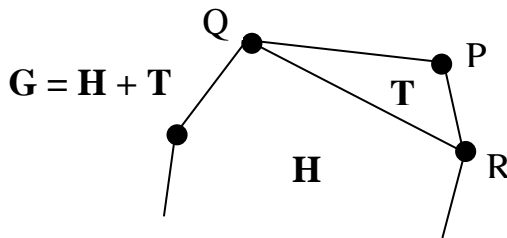
We first check that it works for $n = 3$. This is the familiar case of the angles of a triangle adding up to 180° . We don't prove this here.

Now suppose that the angles of an n -sided polygon total $180n - 360$ degrees.

Let G be a polygon with $n + 1$ sides and let P be any vertex.

Let Q, R be the vertices adjacent to P .

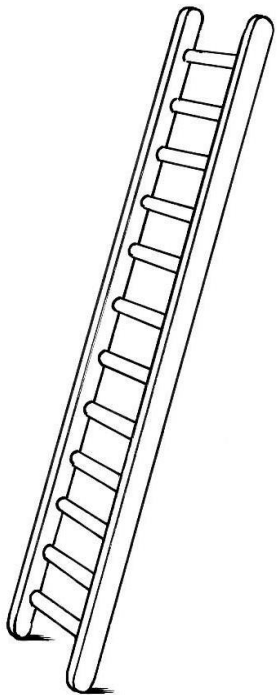
Draw the line QR . This divides the original polygon G into a polygon H with n edges and a triangle T whose vertices are P, Q, R .



The sum of the angles of G is the sum of the angles of H plus the sum of the angles of T .

This is $180n - 360 + 180 = 180(n + 1) - 360$, so the theorem holds for $n + 1$.

Hence, by induction it holds for all n .



The pattern of an induction proof is as follows:

- (1) If $n = \dots$ then so the result holds for $n = \dots$
- (2) Suppose the result holds for n .
- (3) Consider the $n + 1$ case.
- (4) Relate it to the n case.
- (5) Prove it for the $n + 1$ case.
- (6) Therefore, by induction, it is true for all $n \geq \dots$

Don't forget to include the final statement as given in (6).

Now sometimes we can't go from the n case to the $n + 1$ case, but rather we prove the n case on the basis of the truth of the statement for one or more values less than n . This

is called the Strong Form of Induction.

Theorem 2 (Strong Induction): Suppose $P(n)$ is some statement that depends on a positive integer n .

Suppose that if $P(k)$ is true for all $k < n$ then $P(n)$ is true.

Then $P(n)$ is true for all n .

Proof: Let $Q(n)$ be the statement "if $k < n$ then $P(k)$ is true."

$Q(1)$ makes the vacuous claim that "if $k < 1$ then $P(k)$ is true. This is true since there are no such k . (Remember we

are only considering positive integers.) A statement of the form ‘if X then Y’ is always true if X is false.

Suppose $Q(n)$ is true. By the assumption in the statement of the theorem this implies that $P(1), P(2), P(3), \dots, P(n - 1)$ are true. By the assumption in the statement of the theorem this means that $P(n)$ is also true. Hence $Q(n + 1)$ holds. By the simple principle of induction on $Q(n)$, $Q(n)$ holds for all n and so $P(n)$ is true for all n . 🙌😊

Example 2: Prove that if $n > 1$ then $n = p_1 p_2 \dots p_k$ for some primes p_1, p_2, \dots, p_k .

Solution: Suppose this holds for all integers k such that $1 < k < n$.

If n is prime then $n = p_1$ is a prime (in this case $k = 1$).

If n is not prime then $n = ab$ for some integers a, b with $1 < a, b < n$.

By the induction hypothesis $a = p_1 p_2 \dots p_k$ for some k and some primes p_1, p_2, \dots, p_k and

$b = q_1 q_2 \dots q_h$ for some h and some primes q_1, q_2, \dots, q_h .

Hence $n = p_1 p_2 \dots p_k q_1 q_2 \dots q_h$ and so the result holds for n .

§3.2. Inductive Definitions

In chapter 1 we discussed the basic laws of algebra, from which all the properties of the real numbers can be deduced. Can we prove these basic laws? The answer is, we can, provided we define real numbers and their fundamental operations. We sketched how this might be

done, starting with the natural numbers. But can we take it back one step further?

What is the number 2? What does $2 + 2$ mean? And why is it equal to 4? How is multiplication defined? Why is xy always equal to yx ?

The Peano Axioms for the natural numbers (positive integers) set up the natural numbers as a set on which there's an operation $n \rightarrow n^+$. That is, for every natural number n there's a natural number n^+ . You'll soon see that n^+ is just $n + 1$, but until we've defined addition we're not supposed to know that.

The **Peano Axioms** are:

- (1) 0 is a natural number;
- (2) there's no natural number n such that $n^+ = 0$;
- (3) if $m^+ = n^+$ then $m = n$;
- (4) if S is a set of natural numbers that contains 0, and with the property that whenever $n \in S$, so is n^+ , then S is the set of all natural numbers.

Axiom (2) ensures that the negative numbers are not included and axiom (4) ensures that numbers such as $\frac{1}{2}$ are not included. Axiom (4) can also be used to prove the Principle of Induction. The proof we gave for induction was not as rigorous as it might be. This one is.

Theorem 1 again (Principle of Induction):

Suppose $P(n)$ is some statement that includes a variable n .
Suppose that:

- (1) $P(1)$ is true and
- (2) If $P(n)$ is true then $P(n + 1)$ is true.

Then $P(n)$ is true for all positive integers n .

New Proof: Let $S = \{n \mid P(n) \text{ is true}\}$.

Then S contains 0, by (1) and $n \in S$ implies that $n^+ \in S$, by (2).

So by Peano Axiom (4), S is the set of all natural numbers.
In other words $P(n)$ is true for all natural numbers n . 🙌😊

In the context of this description of the natural numbers, we define 1 as 0^+ , $2 = 1^+ = 0^{++}$ and so on. The number 4, for example, can be defined as 0^{++++} .

We define addition inductively as follows:

$$m + 0 = n \text{ for all natural numbers } n;$$

$$m + n^+ \text{ is defined as } (m + n)^+ \text{ for all natural numbers } m, n.$$

$$\begin{aligned} \text{So } 2 + 1 &= 2 + 0^+ = (2 + 0)^+ \\ &= 2^+ \\ &= 3. \end{aligned}$$

$$\begin{aligned} \text{Hence } 2 + 2 &= 2 + 1^+ \\ &= (2 + 1)^+ \\ &= 3^+ \\ &= 4. \end{aligned}$$

At last you've learnt how to *prove* that $2 + 2 = 4$. Of course you'll continue to add numbers the way you

always have without recourse to this rather painful procedure!

Now we'll prove the commutative law for addition, that is $m + n = n + m$. But first we'll have to prove a couple of other results first.

Theorem 3: $m^+ + n = m + n^+$ for all natural numbers m, n .

Proof: We do this by induction on n .

$m^+ + 0 = m^+ = (m + 0)^+ = m + 0^+$ so the theorem holds for 0.

Suppose that $m^+ + n = m + n^+$, that is, suppose the theorem holds for n .

Then $m^+ + n^+ = (m^+ + n)^+$
 $= (m + n^+)^+$ by the induction hypothesis
 $= m + n^{++}$ by the definition of addition,.

So the theorem holds for n^+ , that is, $n + 1$.

Hence by induction it holds for all natural numbers n . 🙌😊

Theorem 4: $0 + n = n$ for all natural numbers n .

Proof: We do this by induction on n .

$0 + 0 = 0$ so the theorem hold for 0.

Suppose that $0 + n = n$, that is, suppose the theorem holds for n .

Then $0 + n^+ = (0 + n)^+ = n^+$ so the theorem hold for n^+ , that is, $n + 1$.

Hence by induction it holds for all natural numbers n . 🙌😊

Theorem 5: $m + n = n + m$ for all natural numbers m, n .

Proof: We do this by induction on n .

$m + 0 = m = 0 + m$ by the above theorem so the present theorem holds for 0.

Suppose that $m + n = n + m$, that is, suppose the theorem holds for n .

Then $m + n^+ = (m + n)^+$ by the definition of addition
= $(n + m)^+$ by the induction hypothesis
= $n + m^+$ by the definition of addition
= $n^+ + m$ by theorem 3

So the present theorem holds for n^+ .

Hence by induction it holds for all natural numbers n . 🙌😊

Multiplication and exponentiation can be defined inductively by:

$$\begin{aligned} m \cdot 0 &= 0; & m^0 &= 1 \\ m \cdot (n + 1) &= mn + m. & m^{n+1} &= m^n. \end{aligned}$$

We can prove all the familiar rules of the arithmetic of natural numbers by induction in a similar way to the way we proved the Commutative Law for Addition.

§3.3. Arithmetic Series

An **arithmetic sequence** is a series in which the same number gets added each time. These are often the basis of ‘guess the next term’ questions.

Example 2: Write down the next term in the sequence 8, 13, 18, 23, ...

Solution: We guess that the pattern is that 5 is added to each term to get the next, in which case the next term will be 28.

Of course there's no guarantee that this is the pattern. It's possible to come up with a very complicated pattern that gives 8, 13, 18, 23 as the first 4 terms but something other than 28 for the fifth. Always with these sorts of questions we have to look for the most obvious pattern.

Example 3: Write down the next term in the sequence 61, 67, 73, 79, ...

Solution: The obvious answer would be 85 because it appears that the rule is to add 6 each time. Wrong! The pattern is to add 5 and then take the next prime number.

Adding 5 to 61 gives 66 and the next prime number after this is 67. This rule produces 61, 67, 73, 79. But if we add 5 to 79 we get 84 and the next prime number after 84 is 89. Of course I've been a bit unfair. The understanding for this type of question is to come up with the simplest rule that fits the numbers that are given and use it to produce the next. Of course there can be disagreement as to what is the simplest rule. If we wanted to use the more complicated rule of adding 5 and going to the next prime we should have said:

What is the next number in the sequence:

61, 67, 73, 79, 89, ...

This would be a fair question, though quite a hard one.

An **arithmetic series** (sometimes called an Arithmetic Progression, or AP for short) is where we add the terms of an arithmetic sequence. If the first term is a and we add d each time then the sequence is

$$a, a + d, a + 2d, a + 3d, \dots$$

and the series is

$$a + (a + d) + (a + 2d) + \dots\dots$$

The number a is called the **first term** and the number d is called the **common difference**. The n 'th term is often denoted by T_n , so $T_1 = a$ and the sum of the first n terms is denoted by S_n .

Theorem 6: For an arithmetic series with first term a , last term L and common difference d :

$$(1) T_n = a + (n - 1)d,$$

$$(2) S_n = \frac{n}{2} [2a + (n - 1)d] .$$

$$(3) S_n = \frac{n}{2} [a + L].$$

Proof: (1) In going from T_1 to T_n there are $n - 1$ steps, each involving the addition of d .

Hence $T_n = a + (n - 1)d$.

$$(2) \text{ Let } T_n = 1 + 2 + \dots + (n-1) + n.$$

$$\text{Then } T_n = n + (n-1) + \dots + 2 + 1.$$

$$\text{Hence } 2T_n = (n+1) + (n+1) + \dots + (n+1) + (n+1) = n(n+1).$$

$$\therefore T_n = \frac{1}{2} n(n+1)$$

$$\begin{aligned} S_n &= na + T_n d \\ &= \frac{1}{2} n(n+1)d + na \\ &= \frac{1}{2} n[2a + (n+1)d]. \end{aligned}$$

$$(3) \text{ Since } L = T_n = a + (n-1)d, S_n = \frac{n}{2} [a + L]. \quad \text{👋😊}$$

Example 4: Find $1 + 2 + 3 + \dots + 100$.

Solution: Here $a = 1$ and $d = 1$.

$$\begin{aligned} \therefore S_{100} &= \frac{100}{2} [1 + 100] \\ &= 50 \times 101 = 5050. \end{aligned}$$

Here the calculation is so simple that one could do it in one's head. There's a story that a young boy called Carl was in a mathematics class where the teacher wanted to occupy the pupils for a whole hour with the minimum amount of work on his part. So he wrote up the above problem, expecting that it would take most of the lesson for the boys to do the 99 additions, while he could snooze. He was most annoyed when Carl called out the answer before the teacher had even sat down! The boys hadn't been taught the arithmetic series formula but Carl had figured it out for himself. He realised that if you add the first and the last you get 101. Similarly if you add the

second and the second last you also get 101. Pairing the 100 numbers into 50 pairs in this way the answer must be 50 times 101.

Of course the teacher wasn't at all pleased and history doesn't record whether Carl was kept in after school for insolence! But he did go on and become the famous mathematician Carl Friedrich Gauss [1777 – 1855].

Example 5: Find the sum of 100 terms of the arithmetic series 5, 12, 19, ...

Solution: Here $a = 5$ and $d = 7$.

$$\begin{aligned}\therefore S_{100} &= \frac{100}{2} [10 + 99.7] \\ &= 50 \times 703 \\ &= 35150.\end{aligned}$$

§3.4. Geometric Series

A **geometric sequence** is one where we multiply each term by the same number to get the next. This number is called the **common ratio**. The sequence can therefore be written as

$$a, ar, ar^2, \dots$$

where a is the first term and r is the common ratio.

A **geometric series** is where we add the terms in a geometric sequence:

$$a + ar + ar^2 + \dots$$

The number a is the **first term**, as before, and the number r is called the **common ratio**. As with arithmetic series we use the symbols T_n and S_n to denote the n 'th term and the sum of the first n terms respectively.

Theorem 7: For a geometric series with first term a and common ratio $r \neq 1$:

$$(1) T_n = ar^{n-1} \text{ and}$$

$$(2) S_n = a \left(\frac{1 - r^n}{1 - r} \right).$$

Proof: (1) From T_1 to T_n there are $n - 1$ steps, each involving the multiplication by r . So in all we are multiplying by r^{n-1} and so $T_n = ar^{n-1}$.

$$(2) S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}.$$

$$\therefore rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n.$$

Subtracting we get $(1 - r)S_n = a - ar^n$.

$$\text{Hence } S_n = a \left(\frac{1 - r^n}{1 - r} \right) = a \left(\frac{r^n - 1}{r - 1} \right). \quad \text{👋😊}$$

The above theorem leaves out what happens when $r = 1$. Clearly with $r - 1$ in the denominator we can't use the above formula for S_n . But if $r = 1$ the series becomes:

$$a + a + a + \dots + a$$

and consequently $S_n = na$ in this case.

Example 6: Find the 20th term and the sum of the first 20 terms of the series $3 + 6 + 12 + \dots$

Solution: $a = 2$, $d = 2$ and $n = 20$.

$T_{100} = 3.2^{19} = 1572864$, with the help of a calculator.

$$S_{20} = 3 \left(\frac{2^{20} - 1}{2 - 1} \right) = 3 \cdot (2^{20} - 1) = 3145725.$$

Example 8: An insect travels 1 metre in the first minute, 50cm in the 2nd and 25cm in the 3rd minute. Each minute it travels half the distance as in the previous minute. How far will it have travelled after 12 minutes?

Solution: Using centimetres as the unit we have $a = 100$ and $r = \frac{1}{2}$.

$$S_{12} = 100 \frac{1 - \frac{1}{2}^{12}}{1 - \frac{1}{2}} = 200 \cdot (1 - \frac{1}{2}^{12}) = 200 \text{ approximately.}$$

It's clear that no matter how much time elapses the insect is never going to get to a point 200 centimetres from the start. It will get closer and closer but never quite reach that point. The distance of 200 centimetres is what is called a 'sum to infinity'.

Now if $r < 1$ we usually write S_n in the form $S_n = a \left(\frac{1 - r^n}{1 - r} \right)$. As n gets larger and larger r^n approaches zero.

So we get what is called the **sum to infinity** of the geometric series:

$$S_{\infty} = a + ar + ar^2 + \dots = \frac{a}{1 - r}.$$

Example 9: Psychologists have determined that each year you live seems shorter than the year before. They estimate that this year will only seem to be 99% as long as last year and next year will appear to be only 99% as long as this

year. Suppose this holds for every year. If you lived forever how long would this eternal life appear to be in terms of the apparent length this year?

Solution: Here we have a geometric series with $a = 1$, $r = 0.99$.

$$S_{\infty} = \frac{1}{0.01} = 100.$$

Example 10: Under the assumptions in exercise 9, assuming that you lived forever when would you appear to have lived for half your life?

Solution: We want to find n so that $S_n = 50$.

$$\therefore \frac{1 - 0.99^n}{.01} = 50.$$

$$\therefore 1 - 0.99^n = 0.5.$$

$$\therefore 0.99^n = 0.5.$$

$$\therefore n = \frac{\log 0.5}{\log 0.99} = 68.96757 \dots$$

So the midpoint of the apparent life of someone who lives forever would appear to be at approximately age 69.

§3.5. Sigma Notation

There is a special notation that we use to write out the sum of a series. If T_n is the n 'th term we can write S_1

$$= T_1 + T_2 + \dots + T_n \text{ as } \sum_{r=1}^n T_r .$$

Example 11: For a geometric series, with first term a and

common ratio r :
$$S_n = \sum_{r=1}^n ar^{n-1} .$$

We can use this notation to express all sorts of series, though whether we can find a formula is another

matter. $\sum_{r=1}^n f(r)$ is defined to be $f(1) + f(2) + \dots + f(n)$.

Example 12:
$$\sum_{r=1}^5 r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Example 13:
$$\sum_{r=1}^6 \frac{1}{r^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} .$$

We can start with some value other than $r = 1$.

Example 12:
$$\sum_{r=0}^4 (r^2 + 1) = (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 35.$$

Example 13:
$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + \dots + n^3.$$

It is possible to express this by a formula that is quicker than having to add all the cubes.

$\sum_{r=1}^n r^3 = \frac{1}{4}[n^2(n+1)^2]$. Although we have not developed

the techniques for finding such formulae we can prove that they are correct by induction.

Theorem 8:
$$\sum_{k=0}^{n-1} \cos k\theta = \frac{\sin\left(\frac{2n-1}{2}\theta\right) + \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}.$$

Proof: $\cos k\theta$ is the real part of $e^{ik\theta}$. Now $\sum_{k=0}^{n-1} (e^{i\theta})^k$ is the

sum to n terms of the GP whose first term is 1 and whose common ratio is $e^{i\theta}$.

The sum of this GP is
$$\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = \frac{(\cos n\theta - 1) + i \sin n\theta}{(\cos \theta - 1) + i \sin \theta}$$

$$= \frac{(\cos n\theta - 1) + i \sin n\theta}{(\cos \theta - 1) + i \sin \theta} \times \frac{(\cos \theta - 1) - i \sin \theta}{(\cos \theta - 1) - i \sin \theta}$$

Thus $\sum_{k=0}^{n-1} \cos k\theta$ is the real part of the above =

$$\frac{(\cos n\theta - 1)(\cos \theta - 1) + \sin n\theta \cdot \sin \theta}{2(1 - \cos \theta)}$$

$$= \frac{\cos n\theta \cdot \cos \theta - \cos \theta - \cos n\theta + 1 + \sin n\theta \cdot \sin \theta}{2(1 - \cos \theta)}$$

$$\begin{aligned}
&= \frac{\cos (n-1)\theta - \cos n\theta + 1 - \cos \theta}{2(1 - \cos \theta)} \\
&= \frac{2\sin \left(\frac{2n-1}{2}\right)\theta \cdot \sin \frac{\theta}{2} + 2\sin^2 \frac{\theta}{2}}{4\sin^2 \frac{\theta}{2}} \\
&= \frac{\sin \left(\frac{2n-1}{2}\right)\theta + \sin \frac{\theta}{2}}{2\sin \frac{\theta}{2}} . \text{ 🙌 😊}
\end{aligned}$$

EXERCISE FOR CHAPTER 3

Exercise 1: Prove by induction on n that $\sum_{k=1}^n \frac{1}{k(k+1)} =$

$$\frac{n}{(n+1)}.$$

Exercise 2: Prove by induction that $\sum_{k=1}^n k^3 = \left(\sum_{k=1}^n k \right)^2$.

Exercise 3: Find $\sum_{k=0}^{n-1} \sin k\theta$.

SOLUTION FOR CHAPTER 3

Exercise 1: It's clearly true for $n = 1$.

Suppose that it is true for n .

$$\begin{aligned} \text{Then } \sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^n \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{(n+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \end{aligned}$$

$$= \frac{n+1}{n+2}, \text{ so it is true for } n+1.$$

Therefore it is true for all n .

If I hadn't insisted that you prove it by induction you might have noticed that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\begin{aligned} \text{so } \sum_{k=1}^n \frac{1}{k(k+1)} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \\ &\quad \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n+1-1}{n+1} \\ &= \frac{n}{n+1}. \end{aligned}$$

Exercise 2: $\sum_{k=1}^n k$ is the sum to n terms of the AP with first term 1 and common difference 1.

$$\text{So } \left(\sum_{k=1}^n k\right)^2 = \frac{1}{4} n^2(n+1)^2.$$

We prove by induction that $\sum_{k=1}^n k^3 = \frac{1}{4} n^2(n+1)^2$.

For $n = 1$ this becomes $1 = 1$, so it is true for $n = 1$.

Suppose that $\sum_{k=1}^n k^3 = \frac{1}{4} n^2(n+1)^2$.

Then $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3$

$$= \frac{1}{4} n^2(n+1)^2 + (n+1)^3$$

$$= \frac{1}{4} (n+1)^2(n^2 + 4n + 4)$$

$$= \frac{1}{4} (n+1)^2(n+2)^2.$$

Hence it is true for $n + 1$.

Therefore, by induction, it is true for all n .

Exercise 3: $\sin k\theta$ is the imaginary of $e^{ik\theta}$. Now $\sum_{k=0}^{n-1} (e^{i\theta})^k$ is

the sum to n terms of the GP whose first term is 1 and whose common ratio is $e^{i\theta}$.

The sum of this GP is $\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = \frac{(\cos n\theta - 1) + i \sin n\theta}{(\cos \theta - 1) + i \sin \theta}$

$$= \frac{(\cos n\theta - 1) + i \sin n\theta}{(\cos \theta - 1) + i \sin \theta} \times \frac{(\cos \theta - 1) - i \sin \theta}{(\cos \theta - 1) - i \sin \theta}$$

$$\begin{aligned}
& \text{Thus } \sum_{k=0}^{n-1} \sin k\theta \text{ is the imaginary part of the above} = \\
& \frac{\sin n\theta \cdot (\cos \theta - 1) - (\cos n\theta - 1) \sin \theta}{2(1 - \cos \theta)} \\
& = \frac{\sin n\theta \cdot \cos \theta - \sin n\theta - \cos n\theta \cdot \sin \theta + \sin \theta}{2(1 - \cos \theta)} \\
& = \frac{\sin (n-1)\theta - \sin n\theta + \sin \theta}{2(1 - \cos \theta)} \\
& = \frac{2\cos\left(\frac{2n-1}{2}\theta\right) \cdot \sin\left(-\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} \\
& = \frac{-\cos\left(\frac{2n-1}{2}\theta\right) + \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}.
\end{aligned}$$